THE SECOND VARIATION OF A DEFINITE INTEGRAL WHEN ONE END-POINT IS VARIABLE*

BY

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The method applied in the following paper to the discussion of the second variation in the case in which one end-point is movable on a fixed curve, is closely analogous to that of WEIERSTRASS † in his treatment of the problem for fixed end-points. The difference arises from the fact that in the present case terms outside of the integral sign must be taken into consideration. As a result of the discussion the analogue of JACOBI's criterion will be derived, defining apparently in a new way the critical point ‡ for the fixed curve along which the end-point varies. The relation between the critical and conjugate points is discussed in § 4.

§ 1. *The expression for the variation of the integral.*

Consider a fixed curve D,

$$x = f(u), \quad y = g(u),$$

and a fixed point $B\ (x_1, y_1)$. Let C be a curve,

$$x = \phi(t), \quad y = \psi(t),$$

cutting D at $A\,(u = u_0,\ t = t_0)$, passing through $B\,(t = t_1)$, and making the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

a minimum with respect to values of the integral taken along other curves joining D and B, and lying in a certain neighborhood of C. The following assumptions are made: §

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† *Lectures on the Calculus of Variations*, 1879.

‡ The same as KNESER's "Brennpunkt." See his *Variationsrechnung*, p. 89.

§ Literal subscripts will be used to denote differentiation, partial when several variables are involved. The zero-subscript or $[\ \ ]_0$ means that in the function designated $t = t_0$, $u = u_0$. Unaccented letters refer to D; while accented letters refer to C.

1) The functions discussed are regular at the points considered;

2) $$[x_u^2 + y_u^2]_0 \neq 0; \quad x'^2 + y'^2 \neq 0, \text{ for } t_0 \leqq t \leqq t_1;$$

3) F satisfies the usual homogeneity condition

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y') \qquad (\kappa > 0); \tag{1}$$

4) $$F(x_0, y_0, x_0', y_0') \neq 0.$$

When the integral is taken along a curve,

$$\bar{x} = \phi(t) + \xi(t), \quad \bar{y} = \psi(t) + \eta(t), \tag{2}$$

the first variation can be put into the well-known form: *

$$\delta I = [F_{x'}\xi + F_{y'}\eta]_{t_0}^{t_1} + \int_{t_0}^{t_1} [G_1\xi + G_2\eta]\,dt, \tag{3}$$

where

$$G_1 = F_x - \frac{d}{dt}F_{x'}, \quad G_2 = F_y - \frac{d}{dt}F_{y'}.$$

According to WEIERSTRASS † the second variation can be expressed in the form:

$$\delta^2 I = [R]_{t_0}^{t_1} + \int_{t_0}^{t_1} [F_1 w'^2 + F_2 w^2]\,dt, \tag{4}$$

where

$$R = L\xi^2 + 2M\xi\eta + N\eta^2, \quad w = y'\xi - x'\eta,$$

the functions F_1, L, M, N, F_2 being defined by the following equations:

$$F_1 = \frac{1}{y'^2}F_{x'x'} = -\frac{1}{x'y'}F_{x'y'} = \frac{1}{x'^2}F_{y'y'},$$

$$\begin{aligned} L &= F_{xx'} - y'y''F_1, \quad N = F_{yy'} - x'x''F_1, \\ M &= F_{x'y} + x''y'F_1 = F_{xy'} + x'y''F_1, \\ F_2 &= \frac{1}{y'^2}(F_{xx} - y''^2F_1 - L') = -\frac{1}{x'y'}(F_{xy} + x''y''F_1 - M') \\ &= \frac{1}{x'^2}(F_{yy} - x''^2F_1 - N'). \end{aligned} \tag{5}$$

In the first place by considering variations of the curve which pass through the end-points A and B considered as fixed, the following two necessary conditions for a minimum are found:

* See KNESER, loc. cit., § 4. The arguments of F and its derivatives are always x, y, x', y'.
† WEIERSTRASS's *Lectures*, 1879.

I. *C must be an extremal* satisfying $G_1 = 0$ and $G_2 = 0$;*

II. *F_1 must be $\geqq 0$ along the arc AB of the curve C.*†

In the second place consider variations which do not pass through A. Inasmuch as only a necessary condition is desired, ξ and η can be chosen in a special manner. Let ξ_0, η_0, ξ, η be defined by the equations:

$$(6)\qquad \begin{aligned} \xi_0 &= f(u_0+\sigma) - f(u_0) = [x_u]_0\sigma + [x_{uu}]_0\frac{\sigma^2}{2} + \cdots, \\ \eta_0 &= g(u_0+\sigma) - g(u_0) = [y_u]_0\sigma + [y_{uu}]_0\frac{\sigma^2}{2} + \cdots, \end{aligned}$$

$$(7)\qquad \begin{aligned} \xi &= \phi_1\xi_0 + \phi_2\eta_0, \\ \eta &= \psi_1\xi_0 + \psi_2\eta_0, \end{aligned}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are functions of t satisfying the relations:

$$\phi_1(t_0) = \psi_2(t_0) = 1, \quad \phi_2(t_0) = \psi_1(t_0) = 0,$$

$$\phi_1(t_1) = \phi_2(t_1) = \psi_1(t_1) = \psi_2(t_1) = 0.$$

A curve (2) constructed with ξ and η as in (7) will be said to *belong to the class* $\bar{C}$. It is evident that each particular curve $\bar{C}$ cuts D when $t = t_0$, and passes through B when $t = t_1$.

For these special variations ΔI can be expressed as a power series in σ, say

$$(8)\qquad \Delta I = S_1\sigma + S_2\frac{\sigma^2}{2} + \cdots.$$

S_1 and S_2 can be calculated from δI and $\delta^2 I$. From (3) and (6), since C passes through B,

$$(9)\qquad \delta I = -[F_{x'}x_u + F_{y'}y_u]_0\sigma - [F_{x'}x_{uu} + F_{y'}y_{uu}]_0\frac{\sigma^2}{2} + \cdots,$$

and therefore from (4) and (9),

$$S_1 = -[F_{x'}x_u + F_{y'}y_u]_0,$$

$$S_2 = -[F_{x'}x_{uu} + F_{y'}y_{uu} + Lx_u^2 + 2Mx_uy_u + Ny_u^2]_0 + \int_{t_0}^{t_1}[F_1\bar{w}'^2 + F_2\bar{w}^2]\,dt,$$

where $\bar{w}$ and $\bar{w}'$ are the coefficients of σ in w and its derivative.

From (8) it follows that a third necessary condition for the existence of a minimum is

III. $$S_1 = 0.$$

* E. g., see KNESER, loc. cit., § 8.

† WEIERSTRASS'S *Lectures*, 1879.

This is the well-known condition for transversality.* It follows also from (8) that *if a minimum exists, S_2 must be* $\geqq 0$ *for all curves of class* $\bar{C}$. The further discussion of S_2 is the principal object of this paper.

§ 2. *A condition which prevents S_2 from becoming negative.*

Suppose now that C satisfies the conditions I and III, and (instead of II) the condition that F_1 is > 0 along the arc AB. Transform (4) by adding with Legendre,

$$0 = -\left[vw^2\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \frac{d(vw^2)}{dt}\, dt.$$

The integrand becomes a homogeneous quadratic expression in w and w'. If for $t_0 \leqq t \leqq t_1$, a regular function v exists satisfying the discriminant relation

$$(v) \qquad v^2 - F_1(F_2 + v') = 0,$$

then $\delta^2 I$ becomes

$$(10) \qquad \delta^2 I = \left[R - vw^2\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} F_1\left[w' + \frac{vw}{F_1}\right]^2 dt.$$

The integral of (v) is expressible in terms of the integral of a linear equation. For when $v = -F_1 U'/U$,

$$(U) \qquad v^2 - F_1(F_2 + v') = \frac{F_1}{U}(F_1 U'' + F_1' U' - F_2 U) = 0.$$

Then

$$(11) \qquad \begin{aligned} S_2 = -\Big[F_{x'} x_{uu} + F_{y'} y_{uu} + L x_u^2 + 2M x_u y_u + N y_u^2 + F_1 \bar{w}^2 \frac{U'}{U} \Big]_0 \\ + \int_{t_0}^{t_1} F_1 \left[\frac{U\bar{w}' - U'\bar{w}}{U}\right]^2 dt. \end{aligned}$$

Assume the general integral of the differential equations $G_1 = 0$ and $G_2 = 0$, which are of the second order, to be

$$x = \phi(t, a, \beta), \qquad y = \psi(t, a, \beta),$$

where a and β are arbitrary constants. Suppose that these equations represent C when $a = \beta = 0$. Then two particular integrals of (U) are †

$$\vartheta_1 = \begin{vmatrix} \phi_t & \phi_a \\ \psi_t & \psi_a \end{vmatrix}, \quad \vartheta_2 = \begin{vmatrix} \phi_t & \phi_\beta \\ \psi_t & \psi_\beta \end{vmatrix},$$

* KNESER, loc. cit., § 10.

† E. g., see WEIERSTRASS's *Lectures*.

where $\phi_t = \phi_t(t, 0, 0)$, etc. Suppose ϑ_1 and ϑ_2 to be linearly independent. Then the general integral of (U) is

$$(12) \qquad U = c_1\vartheta_1 + c_2\vartheta_2.$$

Since ϑ_1 and ϑ_2 are linearly independent they satisfy the equation*

$$(13) \qquad \vartheta_1\vartheta_2' - \vartheta_2\vartheta_1' = \frac{c}{F_1} \qquad (c \neq 0).$$

A particular integral (12) *can now be selected so that in* S_2 *the term outside of the integral vanishes.* Put

$$(14) \qquad \begin{aligned} P &= \left[\frac{F_{x'}x_{uu} + F_{y'}y_{uu}}{x_u^2 + y_u^2}\right]_0 + L_0\cos^2\delta + 2M_0\sin\delta\cos\delta + N_0\sin^2\delta, \\ Q &= \left[F_1\frac{(y'x_u - x'y_u)^2}{x_u^2 + y_u^2}\right]_0 = [F_1]_0(x_0'^2 + y_0'^2)\sin^2(\gamma - \delta), \end{aligned}$$

where γ and δ are the angles at A which C and D make with the x-axis. Then from (11),

$$S_2 = -\left[P + Q\frac{U_0'}{U_0}\right]\left[x_u^2 + y_u^2\right]_0 + \int_{t_0}^{t_1} F_1\left[\frac{U\bar{w}' - U'\bar{w}}{U}\right]^2 dt.$$

KNESER has shown† that if $F \neq 0$ at A, and D cuts C transversally, then D cannot be tangent to C. Therefore Q is $\neq 0$. Since, furthermore, the equation (13) holds when $t = t_0$, c_1 and c_2 can be so determined that

$$(15) \qquad P + Q\frac{U_0'}{U_0} = 0.$$

Two such values are

$$c_1 = P\vartheta_2(t_0) + Q\vartheta_2'(t_0), \quad -c_2 = P\vartheta_1(t_0) + Q\vartheta_1'(t_0).$$

If $\mathrm{H}(t, t_0)$ denotes the particular integral of (U) formed with these constants, then

$$(16) \qquad \mathrm{H}(t, t_0) = P\Theta + Q\frac{\partial\Theta}{\partial t_0},$$

where

$$\Theta(t, t_0) = \begin{vmatrix} \vartheta_1(t) & \vartheta_2(t) \\ \vartheta_1(t_0) & \vartheta_2(t_0) \end{vmatrix}.$$

The integral H is useful in forming a function v to satisfy condition (v), at least when B is near A. For from (13) and (16), when $t = t_0$,

$$\mathrm{H}_0 = Q\frac{c}{[F_1]_0} \neq 0.$$

* See CRAIG, *Linear Differential Equations*, vol. 1, p. 54.

† loc. cit., § 30.

These results lead to the following theorem:

If $\mathrm{H}(t, t_0) \neq 0$ *for* $t_0 \leqq t \leqq t_1$, *then for curves of class* $\bar{C}$, S_2 *can be expressed in the form*

$$S_2 = \int_{t_0}^{t_1} F_1 \left[\frac{U\bar{w}' - U'\bar{w}}{U} \right]^2 dt,$$

which cannot become negative.

§ 3. *The necessary condition.*

By following still more closely the method of Weierstrass it can now be shown that the condition $\mathrm{H}(t, t_0) \neq 0$ $(t_0 \leqq t < t_1)$ is necessary for the existence of a minimum. Suppose that this condition does not hold but that H has a zero t_0' between t_0 and t_1. Then, as will be proved, variations of class $\bar{C}$ can be found which make S_2 and ΔI negative.

Integrate by parts the first term in the integrand of (4). Then

$$\delta^2 I = [R + F_1 w w']_{t_0}^{t_1} - \int_{t_0}^{t_1} w[F_1 w'' + F_1' w' - F_2 w]\, dt.$$

Consider the equation,

$$(U_\epsilon) \qquad F_1 U'' + F_1' U' - (F_2 - \epsilon) U = 0,$$

where ϵ is a constant. From the theory of linear differential equations, an integral H_ϵ of the equation (U_ϵ) exists, depending upon ϵ for its value and having the following properties:

1) It is regular for $t_0 \leqq t \leqq t_1$;

2) $[\mathrm{H}_\epsilon]_0 = \mathrm{H}_0$, $[\mathrm{H}_\epsilon']_0 = \mathrm{H}_0'$;

3) If $\eta > 0$ is selected arbitrarily, $\delta > 0$ can be found such that $|\mathrm{H}_\epsilon - \mathrm{H}| < \eta$ for $t_0 \leqq t \leqq t_1$, if $|\epsilon| < \delta$.

H and H$'$ can not both be zero at t_0'. For otherwise, since the functions involved are regular and $F_1 \neq 0$, the expansion of the left member of (U) could not be identically zero. From 3) therefore, δ can be chosen so small that when $|\epsilon| < \delta$, H_ϵ also vanishes between t_0 and t_1, say at $t_{\epsilon 0}$.

Curves can now be chosen of class $\bar{C}$, such that w satisfies the equation (U_ϵ). For example, let ξ and η be defined for $t_0 \leqq t \leqq t_{\epsilon 0}$ by the equations

$$(17) \qquad \begin{aligned} w = y'\xi - x'\eta &= (y_0'\xi_0 - x_0'\eta_0)\frac{\mathrm{H}_\epsilon}{\mathrm{H}_0}, \\ x'\xi + y'\eta &= (x_0'\xi_0 + y_0'\eta_0)\frac{\mathrm{H}_\epsilon}{\mathrm{H}_0}; \end{aligned}$$

and for $t_{\epsilon 0} \leqq t \leqq t_1$, let $\xi = \eta = 0$. Then

$$(18) \qquad \delta^2 I = [R - F_1 w w']_{t_0}^{t_1} + \epsilon \int_{t_0}^{t_1} w^2\, dt.$$

From (9) and (18) by calculation as before, and since H satisfies (15), it follows that

$$S_2 = \epsilon \int_{t_0}^{t_{\epsilon 0}} \bar{w}^2 \, dt .$$

The function $\bar{w}$ can not be identically zero unless H_ϵ is so; and by 3) H_ϵ can not vanish identically if δ is taken small enough, since H does not. Hence for certain functions ξ, η as in (7), $S_2 \neq 0$ and can be made positive or negative by taking values of ϵ opposite in sign. From § 1 therefore the arc AB can not make I a minimum.

If now the point A' defined on C by t_0' is said to be the *critical point* for the curve D, a fourth necessary condition can be stated as follows:

IV. *If the extremal C, which passes through the fixed point B and cuts the fixed curve D transversally, is to make the integral*

$$I = \int_{t_0}^{t_1} F(x, y, x', y') \, dt$$

a minimum, then B must not lie beyond the critical point defined by D on C; or analytically,

$$\mathrm{H}(t, t_0) \neq 0 \quad \textit{for} \quad t_0 \leqq t < t_1 .$$

§ 4. *Relation between the conjugate and critical points.*

The point conjugate to A is defined* by the zero t_0'' of $\Theta(t, t_0)$, which is nearest to t_0. The functions Θ and H are both integrals of (U) of the form (12), and are linearly independent since $\Theta_0 = 0$ and $\mathrm{H}_0 \neq 0$. By a theorem concerning linear differential equations of the second order † their zeros must separate each other, and $\mathrm{H} = 0$ has therefore one root between t_0 and t_0''.

The expression for H involves the curvature of D at A linearly. The curvature is

$$(19) \qquad \frac{1}{r} = \frac{x_u y_{uu} - x_{uu} y_u}{[x_u^2 + y_u^2]^{\frac{3}{2}}} .$$

By differentiating (1) for κ it is found that

$$x' F_{x'} + y' F_{y'} = F .$$

* WEIERSTRASS'S *Lectures*, and KNESER, loc. cit., §§ 24, 31.

† M. BÔCHER, *An elementary proof of a theorem of Sturm*, Transactions of the American Mathematical Society, vol. 2 (1901), p. 150.

From this equation and III the values of $F_{x'}$ and $F_{y'}$ at A can be determined, and by substitution in (16) H becomes

$$(20)\qquad \mathrm{H}(t, t_0) = \left(\frac{P_1}{r} + P_2\right)\Theta + Q\frac{\partial\Theta}{\partial t_0},$$

where

$$(21)\qquad P_1 = \frac{F_0}{\sqrt{x_0'^2 + y_0'^2}\,\sin(\gamma - \delta)},$$

$$P_2 = L_0 \cos^2\delta + 2M_0 \sin\delta\cos\delta + N_0 \sin^2\delta.$$

Suppose C and A fixed, and D changeable but always transversal to C at A. Then if the expression (20) for H is put equal to zero and solved for r, the resulting function of t will express the value which the radius of curvature of D at A must have in order that t may determine the critical point for D. By the use of (13), (14) and (21) the function and its derivative are found to be

$$r = \frac{-P_1\Theta}{P_2\Theta + Q\dfrac{\partial\Theta}{\partial t_0}},$$

$$\frac{dr}{dt} = -\frac{c^2\sqrt{x_0'^2 + y_0'^2}}{\left[P_2\Theta + Q\dfrac{\partial\Theta}{\partial t_0}\right]^2}\frac{F_0}{F_1}\sin(\gamma - \delta).$$

The denominator of r vanishes once between t_0 and t_0'' for the same reason that H does. From 4) of § 1 the derivative dr/dt is $\neq 0$ and has the sign of

$$\frac{F_0}{F_1}\sin(\gamma - \delta).$$

The radius of curvature (19) is positive when its direction is related to that of the curve D for increasing u as the $+\,y$-axis is to the $+\,x$-axis; otherwise it is negative.

From these results the following theorems can be stated if it is supposed that $F_0 > 0$:

1) *The critical point for a curve D which cuts the extremal C transversally at A, always lies between A and its conjugate A''.*

2) *The position of the critical point is determined by the curvature of D at A.**

3) *If the radius of curvature of D at A is supposed to vary continuously from 0 to ∞ on the same side of D as the arc AB, and from ∞ to 0 on the*

* See KNESER, loc. cit., p. 111.

opposite side, then the critical point moves continuously from A to A'' when there is a minimum, and from A'' to A when there is a maximum.

§ 5. *Relation between the preceding results and those of Kneser. Sufficient conditions.*

KNESER has derived a necessary condition which corresponds to IV. He shows that it is possible to find a set of extremals,*

$$x = \xi(t, a), \quad y = \eta(t, a), \tag{22}$$

each cutting D transversally when $t = t_0$, and giving C for $a = 0$. The curve D is then represented in the vicinity of C by the equations

$$x = \xi(t_0, a), \quad y = \eta(t_0, a),$$

where a is the parameter. The condition III of transversality requires that

$$[F_{x'}\xi_a + F_{y'}\eta_a]_0 = 0$$

for every a near zero, since each curve of the system (22) is transversal to D. This equation can be differentiated for a and the derivatives of F expressed in terms of F_1, L, M, N, from equations (5). From (14) and (20), P and Q depend only upon the curvature and direction of D, and are independent therefore of the parameter representation. It follows that for $a = 0$,

$$\frac{\partial}{\partial a}[F_{x'}\xi_a + F_{y'}\eta_a]_0 = \left[P + Q\frac{\Delta'(0, 0)}{\Delta(0, 0)}\right][\xi_a^2 + \eta_a^2]_0 = 0,$$

where

$$\Delta(t, a) = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix}.$$

$\Delta(t, 0)$ must therefore satisfy (15). It can be proved, as for ϑ_1 and ϑ_2, that $\Delta(t, 0)$ is also an integral of (U). Since both H and $\Delta(t, 0)$ are integrals of (U) satisfying (15) they must be linearly dependent. That is,

$$\mathrm{H}(t, t_0) = C\Delta(t, 0), \quad C \neq 0.$$

The condition IV can therefore be restated in KNESER's form:

IV′. *If C as in* IV *is to make I a minimum, then a necessary condition is*

$$\Delta(t, 0) \neq 0 \textit{ for } t_0 \leqq t < t_1.$$

KNESER proves this condition† by discussing the case in which B coincides

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* KNESER, loc. cit., §30.

† loc. cit., §25.

with the critical point A'. He shows that unless the envelope has a singular point of a particular kind at A', there is no minimum, and so none when B lies beyond A'. The result is a stronger condition than IV′, namely,

$$\Delta(t, 0) \neq 0, \quad \text{for} \quad t_0 \leqq t \leqq t_1. \tag{23}$$

But his proof does not hold if the envelope has the exceptional form mentioned.

The method given in § 3 applies when B lies beyond A', and then includes KNESER's exceptional case. It cannot be used when B and A' coincide. For then it is not certain that the integral H_ϵ can be made to vanish between t_0 and t_1, and since w must vanish for $t = t_1$, the functions ξ, η cannot be constructed as in (17).

If the conditions II and IV are amended to read:

II$_a$. *$F_1 > 0$ for points (x, y) on AB, and for any $(x', y') \neq (0, 0)$,*

IV$_a$. *$H(t, t_0) \neq 0$ for $t_0 \leqq t \leqq t_1$,*

then $\Delta(t, 0)$ satisfies (23). According to KNESER a field can be constructed about AB, and the four conditions I, II$_a$, III, IV$_a$ are sufficient conditions for the arc AB to make the integral a minimum.

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